1 It was developed by Frobenius, Schur which provided tools for Cartan, Weyl's Study.

H. Weyl: The theory of groups and quantum mechanics. for physical importance of the subject.

<u>Remark</u>: Let G_i be two Lie groups. $\phi: G_1 \to G_2$ is a continuous group homomorphism, then ϕ is C^∞.

Not hard to believe. e.g. a linear map is smooth.

Warner, Thm 3.38. 3.39.

We shall focus on the complex representation:

V — a complex vector space of dim r.

A linear representation of G in V is a Lie group homomorphism

$\rho: G \to GL(r, V)$. — general linear transformation of V.

$$\rho(g_1 g_2)(v) = \rho(g_1) \circ \rho(g_2)(v).$$

We often just write $(g_1 \cdot g_2)(v) = g_1(g_2(v))$.

<u>E.g.</u> $Ad: G \to GL(n, \mathfrak{g}_{\mathbb{C}})$. We can complexify $\mathfrak{g}$.

$$g \sim a_g(h) = g \cdot h \cdot g^{-1}$$

$$(a_g)_* := Ad(g). \qquad Ad(g) \in GL(n, \mathfrak{g}).$$

We extend it complex linearly to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.

Lecture of Week 1 & 2 showed that Ad is important in understanding the non-commutativity of G.

Representation is to relate G with a subgroup of $GL(r, \mathbb{C})$.

Defn: (i) ρ is called faithful if $\ker(\rho) = \{e\}$.

(ii) V is called a G-linear space. $g.v = \rho(g)(v)$

(iii): $f: V \to W$ a complex linear map is called a G-map (V, W are G-linear spaces) if

$\rho: G \to GL(r, V)$

$\psi: G \to GL(r', W)$

$$f(\underbrace{g.v}_{\rho(g)(v)}) = g.f(v) \sim \psi(g)(f(v))$$

(iv) $U \subset V$ a subspace is called invariant if, $\forall u \in U$
$g.u \in U$

(v) V is called irreducible if $U = V$ & $U = \{0\}$ are the only two invariant subspaces.

(vi) $\chi_\rho(g) = \text{trace}(\rho(g))$ is called the character function.

(vii) V is called completely reducible if $V = \bigoplus V_i$. $\{V_i\}$ are irreducible G-spaces.

(viii) $\rho_i: G \to V_i$, $i = 1, 2$ two representations are called equivalent if $\exists f: V_1 \to V_2$, G-isomorphism

namely (a) $f: V_1 \to V_2$ is a linear iso. $\iff$ $f^{-1}\rho_2(g)f = \rho_1(g)$

(b) $\underbrace{f(\rho_1(g)v) = \rho_2(g) f(v)}$. In short $f(g\cdot u) = g\cdot f(u)$.

If $\{e_i\}$ is a basis of V_1 $\qquad$ $\{\tilde{e}_s\}$ basis of V_2

$\{e_i^*\}$ is the dual basis $\qquad$ $\Rightarrow$ $\boxed{f(e_i) = \tilde{e}_s f_{si}}$

$\tilde{e}_l = f^{il} f(e_i)$

$f^{-1}(\tilde{e}_l) = f^{il} e_i$

$tr(f^{-1}\rho_2 f) = \sum f^{it}(\rho_2)_{ts} f_{si}$

$\simeq \sum (\rho_2)_{tt}$.

$$f^{-1}\rho_2(g) f(e_i) = f^{-1}(\rho_2)(\tilde{e}_s f_{si})$$
$$= f^{-1}((\rho_2)_{ts}\tilde{e}_t) f_{si}$$
$$= e_r f^{rt} (\rho_2)_{ts} f_{si}$$

$\Rightarrow$ $\chi_{\rho_1}(g) = \text{trace}(\rho_1(g)) = \langle e_i^*, \rho_1(g)(e_i)\rangle$ $\qquad$ $\underline{\rho_1(g) = f^{-1}\rho_2(g) f}$

$$= \langle e_i^*, f^{-1}\rho_2(g) f(e_i)\rangle$$
$$= f^{it}(\rho_2)_{st} f_{si} = \sum [f(\rho_2(g))\cdot f^{-1}]_{ii} = tr(\rho_2(g))$$
$$= \chi_{\rho_2}(g)$$

Namely $\chi_g(g)$ is invariant of the equivalent class of representations. $\qquad tr(ABA^{-1}) = tr(B)$

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2 The key to the theory is the existence of a Haar measure.

Theorem 1: For G a compact Lie group, $\exists\ \mu$ — a measure on G which is bi-invariant, namely invariant under R_g & L_g.

Remark: True for compact topological groups.

See Folland, Prop 11.10 & 11.12.

Pf: On G one may define $\qquad$ at T_eG

$$\omega_g((X_1)_g \ldots (X_n)_g) \doteq \omega((L_{g^{-1}})_* X_1, \ldots (L_{g^{-1}})_* X_n)$$

$\{X_i\}$ X_i tangent at T_gG

for ω a fixed n-form at e.

Namely we may write $\omega = X_1^* \wedge \cdots \wedge X_n^*$

if $X_1 \dots X_n$ are left invariant vector fields.

Such ω is denoted as ω^L – namely left invariant (G) n-form.

$$(L_g)^* \omega^L = \omega^L$$

$\forall\ \omega'$ another left invariant n-form $\underbrace{\omega' = c\,\omega^L}$ for some $c \neq 0$.

Same construction works for ω^R – a right invariant n-form.

$$\omega^R(g) = \delta(g)\,\omega^L(g)$$

Now $\omega^R = \delta\,\omega^L$ for some δ – a non-vanishing function.

$R_g L_x = L_x R_g \Rightarrow L_x^* R_g^* = R_g^* L_x^* \Rightarrow L_x^*(R_g^* \omega^L) = R_g^* \omega^L \Rightarrow$

It can be shown $\underbrace{\delta(gh) = \delta(g)\,\delta(h)}$.

$R_g^* \omega^L = c(g)\,\omega^L$

$\shortparallel$

$R_g^*(\delta^{-1}\omega^R) = c(g)\,\delta^{-1}\omega^R$

$\Rightarrow \delta^{-1}(xg) = c(g)\,\delta^{-1}(x)$

$\Rightarrow c(g) = \delta(x)\,\delta^{-1}(xg)$

$x = e$

$c(g) = \delta^{-1}(g)$

Namely $\delta : G \to \mathbb{R}^*$ is a homomorphism

$$\boxed{\delta(e) = 1}$$

The $\underbrace{\text{compactness of } G}$ $\Rightarrow$ $\delta \equiv 1$.

Hence ω^L is also Right-invariant, which provides a bi-invariant measure on G.

δ is called "modular function".

We normalize $\int_G d\mu = 1$

$\forall f$

$$\int_G f(g)\,\omega_g \doteq \int_G f(g)\,d\mu(g)$$

Corollary (a) Compact Lie group G admits a bi-invariant Riemannian metric

(b) If V is a real/complex G-space, G is compact then $\exists \langle\ ,\ \rangle$ on V which is G-invariant, i.e.
(real or Hermitian)
$$\langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle$$

Pf: We only need (b).

$\langle , \rangle_0$ any metric on V

$$\langle v_1, v_2\rangle := \int_G \langle gv_1, gv_2\rangle_0 \, d\mu(g)$$

$g' = g\sigma$

$$\langle \sigma v_1, \sigma v_2\rangle = \int_G \langle g\sigma v_1, g\sigma v_2\rangle_0 \, d\mu(g)$$

g' $\qquad g = g'\sigma^{-1} = (R_{\sigma^{-1}})(g')$

$$= \int_G \langle g'v_1, g'v_2\rangle_0 \, d\mu(g') = \langle v_1, v_2\rangle.$$ $\square$

This makes the representation orthogonal/unitary.

The main consequence:

Thm2: Any real/complex representation of G, a compact Lie group is completely reducible.

(3) Character functions & irreducibility.

Schur Lemma: Let V, W be two irreducible G-spaces. (non-zero).
Let $A: V \to W$ be a G-linear map. Then
either A is invertible or $A \equiv 0$.

Pf. Consider $\ker A$ & $\operatorname{Im} A$. $\square$

$v \in \ker A$
$A(gv) = g A(v)$ $\quad (A(v) = 0)$
$\Rightarrow gv \in \ker(A)$

Coro: $W = V$ & a complex G-space

$A = \lambda I \quad \lambda \in \mathbb{C}$.

Pf: $A - \lambda_0 id$ is a G-linear map.

λ_0 is an eigenvalue $\Rightarrow$ $A - \lambda_0 id \equiv 0$. $\square$

Theorem 3 (i) Two (complex) representations of ρ & ψ: $G \to GL(r, \mathbb{C})$ are equivalent iff $\chi_\rho = \chi_\psi$

(ii) A (complex) representation $\rho: G \to GL(r, V)$ is irreducible iff $\int_G \chi_\rho(g)\, \overline{\chi_\rho(g)}\, d\mu(g) = 1$.

$\underset{\uparrow}{d\mu}$ Haar measure

$1 = \int_G d\mu(g)$.

Remarks: (a) χ is constant in a conjugacy class.

$$\chi_\rho(\sigma g \sigma^{-1}) = tr\left(\rho(\sigma)\, \rho(g)\, \rho(\sigma)^{-1} \right) = tr\,(\rho(g)) = \chi_\rho(g)$$

(b) $\rho_1 \oplus \rho_2$: $\quad \rho_i : G \to GL(r_i, V_i)$

$\chi_{\rho_1 + \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

$$(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g)$$

$$\begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(c) $\rho_1 \otimes \rho_2$: $G \to V_1 \otimes V_2$

$$(\rho_1 \otimes \rho_2(g))(v_1 \otimes v_2) = \underbrace{\rho_1(g)\, v_1}_{\in V_1} \otimes \underbrace{\rho_2(g)\, v_2}_{\in V_2}$$

$\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1} \cdot \chi_{\rho_2}$ (Check!).

[Linear representation of finite groups. Serre]

Function (Continuous) χ on G satisfies $\chi(g\sigma g^{-1}) = \chi(\sigma)$ is called a class function. $Cl(G)$ denotes the total.

$K_c(G)$ — representation ring (w. r. t the addition & product)

$$\chi : K_c(G) \to Cl(G) \quad \text{is a ring homomorphism.}$$

Since the computation involves trace, pick $\{e_i\}$, unitary basis of V

$\{\tilde{e}_s\}$ — of W.

$$\phi(g)(e_i) = e_j \phi_{ji} \qquad (e_1 \dots e_n)\cdot(\phi_{ji})$$

$$\phi(g)^{-1}(e_i) = \overline{\phi(g)}^{tr}(e_i) = e_j \overline{\phi_{ij}}$$

Now let $\boxed{E_{ab}}: V \to W$

$$E_{ab}(e_i) = \begin{cases} 0 & \text{if } b \neq i \\ \tilde{e}_a & \text{if } b = i \end{cases}$$

Consider Case 1: ψ, ϕ are NOT equivalent

$$0 = \int_G \psi(g) \cdot E_{ab} \cdot \phi(g^{-1}) \, d\mu(g) \qquad \text{by Schur's Lemma}$$

G-linear map

check: $\bar{A} = \int_G \psi(g) A \phi(g^{-1}) d\mu(g)$

$\bar{A}(\sigma v) = \sigma \bar{A}(v)$

Precisely $\bar{A}(\phi(\sigma)) = \psi(\sigma) \bar{A}(v)$

$$\Rightarrow \int_G \psi(g) \cdot E_{ab} \cdot \phi(g^{-1})(e_i)$$

$$= \int \psi(g) \cdot E_{ab} \cdot (e_j) \overline{\phi_{ij}}$$

$$= \int \delta_{bj} \, \psi(g)(\tilde{e}_a) \overline{\phi_{ij}} = \sum_s \tilde{e}_s \int \psi_{sa} \overline{\phi_{ij}} \delta_{bj}$$

$$= \tilde{e}_s \int \psi_{sa} \overline{\phi_{ib}}$$

(*1) $\Rightarrow \boxed{\int_G \psi_{sa}(g)\overline{\phi_{ib}(g)}\, d\mu(g) = 0}$ $\quad \forall \quad 1 \le i, b \le n$, $\; 1 \le s, a \le m$

$\chi_\psi(g) = \sum_{i=1}^{m} \psi_{ii}(g)$

$$\Rightarrow \underbrace{\int_G \chi_\psi \cdot \overline{\chi_\phi} = 0}\qquad \int_G \psi_{ss}(g)\,\overline{\phi_{ii}(g)}\, d\mu(g)$$

Namely χ_ϕ, χ_ψ are orthogonal to each other if they are NOT equivalent.

Case 2: $\psi = \phi \Longrightarrow$

$$\int_G \underbrace{\phi(g)\boxed{E_{ab}}\phi^{-1}(g)}$$ is a multiple of id

i.e. $= \lambda_{ab}\, id$ $\qquad \lambda_{a,b} \dim(V) = \int_G tr(E_{ab})$

Taking trace $\Rightarrow$ $\quad a \ne b \quad \boxed{\lambda_{ab} = 0}$

$$\lambda_{aa} \dim V = \int_G tr(E_{aa})\, d\mu(g) = 1$$

$$\Rightarrow \boxed{\lambda_{aa} = \frac{1}{\dim V.}}$$

Similarly. $\quad \frac{\delta_{ab}}{\dim V}\, id = \int \phi(g)\, E_{ab}\, \phi^{-1}(g)$

$$\Rightarrow \frac{\delta_{ab}}{\dim V}(e_i) = \sum_s e_s \int \phi_{sa}\,\overline{\phi_{ib}}$$

(*2) $\Rightarrow \boxed{\int \phi_{sa}\,\overline{\phi_{ib}} = \frac{\delta_{si}\,\delta_{ab}}{\dim V.}}$

$a=s \qquad i$

$$\int \chi_\phi \overline{\chi_\phi} = \int \phi_{ss} \overline{\phi_{ii}}$$

$$= \sum_{s,i} \frac{\delta_{si}\, \delta_{si}}{\dim V} = 1.$$

Now we may derive the rest of Theorem.